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Interiors with relativistic dust flow

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Abstract. New dust interiors admitting three Killing vectors are derived and matched to vacuum exteriors, and some of their properties are investigated.

1. Introduction

This paper is a continuation of earlier studies by Hansen and Winicour (1975), Winicour (1975), Vishveshwara and Winicour (1977) and Hoenselaers and Vishveshwara (1978) on interior solutions admitting an Abelian orthogonally transitive timelike G_2.

In the previous work the quantity $\lambda_{01}^2-\lambda_{11}\lambda_{00}=2\tau^2$ was always assumed to be variable and was used as a harmonic coordinate in the case of dust solutions, as it satisfied a two-dimensional Laplace equation. However, here we shall take τ as a constant.

The relevant field equations and the notation to which we shall adhere throughout were derived in the papers cited above; in particular we shall refer to the last two of them as I and II respectively.

2. The solution

First it may be observed that nothing prevents us from assuming (without loss of generality)

$$\tau=1 \tag{2.1}$$

in the equations derived in II. On the other hand equation (II 2.9)

$$D^aD_a\tau=16\pi\tau p$$

implies

$$p=0 \tag{2.2}$$

and we are thus dealing with matter consisting of dust. The other equations can be combined to yield

$$D^m\left[\frac{1}{2\eta\psi}D_m\psi-\frac{\psi}{\eta}D_m\left(\frac{\eta^2}{\psi}\right)\right]=0 \tag{2.3}$$

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$$16\pi\frac{\eta^2}{\psi^2}\mu = \mathrm{D}^m\left(\frac{\eta^2}{\psi}\right)\mathrm{D}_m\frac{\eta^2}{\psi} - \frac{1}{4\psi^4}\mathrm{D}_m\psi\mathrm{D}^m\psi. \tag{2.4}$$

Now one has to select a state function relating ψ and η. Obviously the easiest choice is to take one of them to be constant. Furthermore we assume all quantities to depend only on one of the coordinates.

For $\eta = \text{const}$ the solution of (2.3) would be $\psi = -\mathrm{e}^{ax}$, but from equation (2.7) below it can be shown that λ_{11} would be strictly negative. We thus discard this case and assume

$$\psi = \text{const}. \tag{2.5}$$

This gives

$$\begin{aligned} &\eta = ax \\ &\mu\, \mathrm{e}^{2\phi} = (1/4\pi)a^2 \\ &\Omega = \text{const} \end{aligned} \tag{2.6}$$

and the λ_α to be calculated from (II 2.17)

$$\begin{aligned} &\lambda_{01} = (\Omega/\psi)(\tfrac{1}{2} - \eta^2) + \eta \\ &\lambda_{11} = (1/\Omega)(\eta - \lambda_{01}) \\ &\lambda_{00} = \psi - \Omega(\lambda_{01} + \eta) \end{aligned} \tag{2.7}$$

read

$$\begin{aligned} &\lambda_{01} = (\Omega/\psi)(\tfrac{1}{2} - a^2x^2) + ax \\ &\lambda_{11} = (1/\psi)(a^2x^2 - \tfrac{1}{2}) \\ &\lambda_{00} = \psi\{[1-(a\Omega/\psi)x]^2 - \Omega^2/2\psi^2\}. \end{aligned} \tag{2.8}$$

The remaining equation for $\mathrm{e}^{2\phi}$ is (II 2.3).

$$R = -\mathrm{e}^{-2\phi}(\partial_x^2\phi + \partial_y^2\phi) = \tfrac{1}{2}\mathrm{D}^m\lambda^\alpha\mathrm{D}_m\lambda_\alpha + 8\pi\mu. \tag{2.9}$$

It surprisingly turns out, as $\lambda'_\alpha\lambda^{\alpha'} = -2a^2$ $(' = \partial_x)$, that $R \equiv 0$ and hence

$$\phi = cx + d. \tag{2.10}$$

3. The exterior solution and matching

The equations and a convenient method of deriving the exterior solution, which apply also in the case $\tau = 1$, have been given in I. Here we just list the results.

$$\begin{aligned} &\lambda_\alpha = A_\alpha\,\mathrm{e}^{bx} + B_\alpha\,\mathrm{e}^{-bx}, \qquad b^2 \neq 0 \\ &A_\alpha A^\alpha = B_\alpha B^\alpha = A_\alpha B^\alpha + \tfrac{1}{2} = 0 \end{aligned} \tag{3.1a}$$

$$\begin{aligned} &\lambda_x = A_\alpha x + B_\alpha, \qquad b = 0 \\ &A_\alpha A^\alpha = A_\alpha B^\alpha = B_\alpha B^\alpha + 1 = 0. \end{aligned} \tag{3.1b}$$

In the case $b^2 < 0$ in (3.1a) the solution for real λ_α is

$$\begin{aligned} &\lambda_\alpha = A_\alpha \sin(bx) + B_\alpha \cos(bx) \\ &A_\alpha A^\alpha = B_\alpha B^\alpha = -1, \qquad A_\alpha B^\alpha = 0. \end{aligned} \tag{3.1c}$$

In all three cases ϕ is given by

$$\phi = \mp b^2x^2/4 + \alpha x + \beta \tag{3.2}$$

where the $\mp$ signs stand for the cases (3.1a) and (3.1c) respectively.

The homogeneous part of ϕ as a solution of (2.9) has an intimate connection with the orbits of the third, i.e. ∂_y, Killing vector present in our solution. It also has consequences for the matching. We shall discuss this later.

For matching the interior solution derived in § 2 to the exterior one solves

$$\lambda^{\mathrm{i}}_{\alpha 0} = \lambda^{\mathrm{e}}_{\alpha 0}, \qquad \lambda'^{\mathrm{i}}_{\alpha 0} = \lambda'^{\mathrm{e}}_{\alpha 0}$$

where i and e stand for the interior and exterior respectively. The subscript 0 indicates that the value is to be taken at the boundary $x = x_0$.

Calculations using (3.1c) give

$$\begin{aligned} &b^2 = 2a^2 \\ &A_\alpha = (1/b)(b\lambda_\alpha \sin(bx) + \lambda'_\alpha \cos(bx))_0 \\ &B_\alpha = (1/b)(b\lambda_\alpha \cos(bx) - \lambda'_\alpha \sin(bx))_0. \end{aligned} \tag{3.3}$$

For the matching of ϕ one has to distinguish between two cases, depending on whether the homogeneous parts of (2.10) and (3.2) are included or not.

Case I: $c = 1, d = 0$. Matching of ϕ and ϕ' gives

$$x_0 = a^2x_0^2/2 + \alpha x_0 + \beta, \qquad 1 = a^2x_0 + \alpha$$

or

$$\alpha = 1 - a^2x_0, \qquad \beta = a^2x_0^2/2. \tag{3.4}$$

Case II: $c = d = \alpha = 0$. The matching conditions are now

$$0 = a^2x_0^2/2 + \beta, \qquad 0 = a^2x_0$$

which shows that a continuous first derivative is only possible for $x_0 = 0$. Otherwise we have to accept a 'kink' in ϕ which leads to a surface stress–energy tensor. In any case

$$\beta = -a^2x_0^2/2. \tag{3.5}$$

4. Discussion

As has been mentioned before, the homogeneous part of the solution of (2.9) is connected with the structure of the orbits of the ∂_y Killing vector. Consider case I: the norm of the ∂_y Killing vector is e^{2x}. Identifying the points y and $y = 2\pi$ one sees that the Killing vector has closed orbits and the axis is approached as $x \to -\infty$. On the other hand, in case II the norm is 1 and hence the Killing vector is a translation.

However, in neither case can we use our interior solution for all x. From (2.8) it is obvious that various pathologies appear if x does not satisfy

$$\begin{aligned} &x \in (-1/\sqrt{2}a, 1/\sqrt{2}a) \\ &x \notin (-1/\sqrt{2}a + \psi/a\Omega, 1/\sqrt{2}a + \psi/a\Omega) \end{aligned}$$

which means that x is bounded by

$$\begin{aligned} &-1/\sqrt{2}a < x < -1/\sqrt{2}a + \psi/a\Omega, && 0 < \psi a\Omega < \sqrt{2}/a \\ &1/\sqrt{2}a + \psi/a\Omega < x < 1/\sqrt{2}a, && -\sqrt{2}/a < \psi/a\Omega < 0 \\ &-1/\sqrt{2}a < x < 1/\sqrt{2}a, && |\psi/a\Omega| > \sqrt{2}/a. \end{aligned} \tag{4.1}$$

While the matching can always be done in case I, case II presents the problem that a continuous first derivative of ϕ can be obtained only at $x_0 = 0$. Let x_1 denote the other matching surface. The calculation of the surface stress–energy tensor according to well known methods (cf. II) yields as non-vanishing components

$$S_3^3 = S_4^4 = (1/8\pi)a^2 x_1 \exp(a^2 x_1^2). \tag{4.2}$$

The four-velocity of the matter constituting the surface, i.e. the timelike eigenvector of S_α^β, is not uniquely determined; however, the density and the mean pressure are given by

$$\mu_s = -p_s = -S_3^3 \tag{4.3}$$

from which one concludes that x_1 has to be less than zero. Of course, this gives conditions on a, ψ and Ω according to (4.1).

The total mass and momentum can be calculated from the Komar integrals as given by Hansen and Winicour. One finds

$$\begin{aligned} m &= (a^2/2)[x_0 - x_1 - (a\Omega/\psi)(x_0^2 - x_1^2)] \\ j &= -(a^3/4\psi)(x_0^2 - x_1^2), \qquad x_0 > x_1. \end{aligned} \tag{4.4}$$

For the two matching surfaces x_0 and x_1 within the limits imposed by (4.1) one can show that m is positive. There is, however, the surprising result that, for $x_1 = -x_0$, j becomes zero. To analyse this behaviour we consider the trajectory of the locally non-rotating observer

$$\zeta^\alpha = \xi_0^\alpha - (\lambda_{01}/\lambda_{11})\xi_1^\alpha, \qquad \zeta_\alpha \zeta^\alpha = -2/\lambda_{11} \tag{4.5}$$

and the part of the four-velocity orthogonal to it

$$_\perp u^\alpha = u^\beta h_\beta^\alpha = (1/\sqrt{-\psi})(\xi_0^\beta + \Omega \xi_1^\beta)[\delta_\beta^\alpha + (\lambda_{11}/2)\zeta_\beta \zeta^\alpha].$$

One finds

$$_\perp u^\alpha = (1/\sqrt{-\psi}\lambda_{11})\xi_1^\alpha(\lambda_{01} + \Omega\lambda_{11}) = (1/\sqrt{-\psi}\lambda_{11})\xi_1^\alpha a x. \tag{4.6}$$

The interpretation is that the flow of matter changes direction at $x = 0$. Thus the above result becomes understandable.

The total space–time consists of three regions in each of which the metric reads

$$ds^2 = e^{2\phi}(dx^2 + dy^2) + \lambda_{11}\, dz^2 + 2\lambda_{01}\, dz\, dt + \lambda_{11}\, dt^2.$$

The region in which the dust is present, and for which ϕ and the λ's are given by (2.10) and (2.8) respectively, has on each side a vacuum region whose ϕ and λ's can be found in (3.2) and (3.1c). The various constants appearing in those expressions are related by the matching conditions (3.3) and (3.4) respectively and (3.5).

To summarise, the model of case I has to be interpreted as a hollow cylinder in which matter flows up and/or down. Mass and momentum per unit height are given by (4.4).

Case II is an infinite slab in which matter flows in one direction over a surface whose pressure and density are given by (4.3). To find the mass and momentum per unit area one has to divide the expressions (4.4) by 2π.

References

Hansen R and Winicour J 1975 *J. Math. Phys.* **16** 804
Hoenselaers C and Vishveshwara C V 1978 *Gen. Rel. Grav.*
Vishveshwara C V and Winicour J 1977 *J. Math. Phys.* **18** 1280
Winicour J 1975 *J. Math. Phys.* **16** 1806